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VISCOPLASTIC DEFORMATION OF ANNULAR PLATES

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Viscoplasticity is one of the most reliable and convenient methods of taking account of the dependence of the strength properties of materials on the loading rate [1, 2]. Analytic solutions of problems of quasistatic loading of sufficiently complex structure elements, which are convenient to obtain by linearizing the fundamental nonlinear viscoplasticity relationships, are of significant interest for practice.

This paper illustrates the utilization of one of the possible linearization methods. The solutions obtained for hinge-supported and clamped annular plates satisfy both the kinematic conditions and the equilibrium equations exactly.

1. A generalization of the simplest dependences for a stiffly viscoplastic material is presented in [1] and reduces to a dynamic flow criterion of the form

$$\sqrt{J_2} = k\left[1 + \Phi^{-1}\left(\frac{\sqrt{I_2}}{\gamma}\right)\right], \tag{1.1}$$

where k is the shear yield point, J_2, I_2 are the second invariants of the stress and strain rate deviators, γ is a coefficient characterizing the ratio between the viscous and plastic properties of the material, Φ is the symbol for a certain function, and Φ^{-1} is the symbol of the reciprocal function.

The associated flow law remains valid. The nonlinear Mises condition is used here as the initial flow condition in stresses. The radius of the circular cylindrical flow surface in the space of the principal stresses is determined also by a nonlinear combination of the principal strain rates. It is easy to see that points of the ellipse (Fig. 1) in the plane of the principal strain rates $\varepsilon_1 - \varepsilon_2$ correspond to points lying on an ellipse similar to the Mises ellipse in the plane of the principal stresses $\sigma_1 - \sigma_2$ for the plane stress state of an incompressible material. To linearize the initial nonlinear relationship it is sufficient to replace the ellipses by certain similar polygons by conserving the similarity of such polygons as the sizes change. For instance, if the ellipse J_2 = const is replaced by the hexagon 1 (Fig. 1a), similar to the Tresk hexagon, then by replacing the ellipse I_2 = const by hexagons 1 or 2 (Fig. 1b), we obtain the relationships, respectively, for the linear function F

$$\max(\sigma_\alpha - \sigma_\beta) = \sigma_T + \mu \max|\varepsilon_\gamma|, \quad \max(\sigma_\alpha - \sigma_\beta) = \sigma_T + (1/2)\mu|\varepsilon_\alpha - \varepsilon_\beta|, \tag{1.2}$$

where the subscripts α, β correspond to the maximal and minimal values of the quantities; ε_γ is the maximal strain rate in absolute value, and $\mu = 3k/2\gamma$ is the viscosity coefficient

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determined experimentally in the uniaxial stress state.

Such a linearization method, in contrast to that proposed earlier by Prager [2], retains the isotropy of reinforcement associated with the strain rate assumed by the initial relationship (1.1). However, it should be noted that the initial relationship possesses still another remarkable property, namely: Not only is the normal to the instantaneous flow surface in agreement in direction with the deformation vector, but also the normal to the curve I_2 = const agrees with the stress vector, i.e., this curve is a surface of the dissipative function level [3]. Only the relation (1.2) (hexagon 1, Fig. 1b) possesses such a property for the initial Tresk flow condition (hexagon 2, Fig. 1a), and only the hexagon 2 (Fig. 1b) possesses this property for the initial flow condition of the maximal reduced stress (hexagon 2, Fig. 1a).

The relationship (1.2) will be used below. For plates subjected to axisymmetric bending, we can obtain [4] in this case

$$\max|m_\alpha - m_\beta| = 1 + \nu|k_\gamma|, \tag{1.3}$$

where $m_1 = M_1/M_0$, $m_2 = M_2/M_0$, M_1, M_2 are the radial and circumferential bending moments, $M_0 = \sigma_T h^2$, $m_3 = 0$, $k_3 = -(k_1 + k_2)$, k_1 and k_2 are the rates of change of the dimensionless curvatures, $\nu = 2\mu h/(3\sigma_T R)$; and R, 2h are the external radius and thickness of the plate.

The rates of curvature are determined by the expressions

$$k_1 = -\frac{d^2w}{d\rho^2}, \quad k_2 = -\frac{1}{\rho}\frac{dw}{d\rho}, \tag{1.4}$$

where w, ρ are the deflection rate and the running radius referred to the radius R.

The problem of deformation of a hinge-supported circular plate subjected to uniform pressure for different kinds of dependences between the stress and the strain rates is examined in [1, 2, 4, 5]. The problem of viscoplastic deformation of annular plates is solved below.

2. Let a hinge-supported plate with an orifice of radius a be loaded along the edge of the orifice by a force P. Using the equilibrium equation in the form

$$d(\rho m_1)/d\rho - m_2 = -p \quad (p = p/(2\pi M_T)) \tag{2.1}$$

or

$$dm_1/d\rho + (m_1 - m_2)/\rho = -p/\rho, \tag{2.2}$$

the boundary conditions

$$m_1 = 0 \text{ when } \rho = \xi = a/R, \ \rho = 1 \tag{2.3}$$

and (1.3) and (1.4), it can be shown that the point J (see Fig. 1a) cannot be taken as the flow condition in any finite interval of variation of ρ and it is impossible to construct a statistically allowable field of internal forces in the case under consideration by using the line JN. Indeed, the conditions $m_1 = 0$, $m_2 = 1 + \nu k_2$ resulting sequentially in the expressions $m_2 = p$, $k_2 = k_1 = (p - 1)/\nu$ are satisfied for the point J. But in conformity with the associated flow law, the inequalities $-k_2 \leqslant k_1 \leqslant 0$ that are not compatible with the expressions written above, should be satisfied.

If the line JN is taken as the flow condition, then the condition $k_1 = -k_2$ (the point A, Fig. 1b) resulting in a value of the circumferential curvature rate $k_2 = C/\rho^2$ should be satisfied, where C is the constant of integration.

The flow condition $m_1 - m_2 = 1 + \nu k_2$, the equilibrium equation (2.2), and the boundary conditions (2.3) result in the expression

$$m_1 = (p-1)\left[\frac{\xi^2 \ln \xi (1-\rho^2)}{\rho^2(1-\xi^2)} - \ln \rho\right].$$

It is easy to show that the moment m_1 is positive for $\rho < 1$, which contradicts the flow condition JN.

The flow condition corresponding to the line JS

$$m_2 = 1 + \nu k_2, \quad m_1 > 0, \quad m_1 < m_2 \tag{2.4}$$

and its associated condition for the curvature (the point B, Fig. 1b) $k_1 = 0$ permit finding

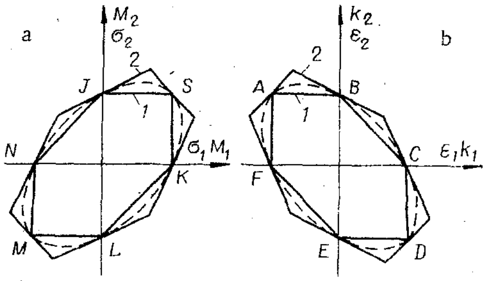

Fig. 1

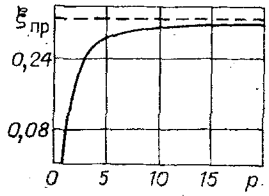

Fig. 2

$$w = w_0(1-\rho)/(1-\xi),\ \nu w_0 = -(p-1)(1-\xi)^2/\ln\xi,\ m_1 = -(p-1)[(1-\xi)\ln\rho - (1-\rho)\ln\xi]/(\rho\ln\xi), \tag{2.5}$$

by using the expressions (1.4) and (2.1) and the boundary conditions (2.3), where w_0 is the rate of deflection of the orifice edge.

It follows from the above that the plate surface becomes conical, while the deflection rate is proportional to the difference between the acting load and the limit force for the plate of stiffly plastic material. Conservation of the flow conditions for the solution constructed is due to satisfaction of the inequalities (2.4). Using (2.5), we find that the extremal value is $m_1 = (p - 1)(e^x/ex - 1)$ $[x = \ln\xi/(\xi - 1)]$.

It can be shown that for any value of ξ in the segment (0, 1), the inequalities $x > 1$, $e^x/ex > 1$ will be satisfied, i.e., the extremal value m_1 is always positive. The uniqueness of the extremum point and the boundary conditions (2.3) afford the possibility of making the deduction that the first of the inequalities (2.4) is always satisfied. To determine the conditions for satisfying the second of the inequalities (2.4), we consider the quantity

$$m_2 - m_1 = 1 + (p-1)\left[\frac{(1-\xi)(\ln\rho - 1)}{\rho\ln\xi} + 1 - \frac{1}{\rho}\right].$$

Since the second term in the right side can be as close to 1 as desired for large p, we investigate the expression in the square brackets. Its extremal value is $f_1 = (x - e^{x-2})/x$, where the extremum point for $\xi < 0.203$ is in the interval $[\xi, 1]$. The function f_1 is nonnegative in the interval of x values [0.1587; 3.1462]. For the relative orifice radius this corresponds to the condition

$$\xi \geqslant \xi_1 = 0.0504. \tag{2.6}$$

On the basis of the above, the deduction can be made that the solution (2.5) obtained is statistically and kinematically allowable for all values of ξ satisfying condition (2.6).

For an annular plate loaded by a uniformly distributed load q, by solving the equilibrium equation

$$\frac{dm_1}{d\rho} + \frac{m_1 - m_2}{\rho} = -\frac{q_1\rho}{2} + \frac{q_1\xi^2}{2\rho},\quad q_1 = \frac{qR^2}{M_T}$$

for the same boundary condition (2.3) and the flow condition (2.4), we can obtain

$$w = \frac{w_0(1-\rho)}{1-\xi},\quad \nu w_0 = \left[\frac{6}{(1-\xi)(1+2\xi)} - q_1\right]\frac{(1-\xi)^3(1+2\xi)}{6\ln\xi}. \tag{2.7}$$

An analysis analogous to that performed above shows that the first inequality from (2.4) is always satisfied, while the second is satisfied for any values of q under the condition $\xi > 0.1233$, i.e., the solution (2.7) obtained is applicable for sufficiently small values of the relative orifice.

3. Let a plate, clamped stiffly along the outer edge, be loaded by a force P distributed along the edge of the orifice. Evidently

$$w = 0,\ dw/d\rho = 0. \tag{3.1}$$

should be taken as kinematic boundary conditions on the edge $\rho = 1$.

This last condition is related to the evident impossibility of the formation of plastic hinges in a viscoplastic body. It follows from this condition that the curvature is $k_2 = 0$ for $\rho = 1$ and the flow condition at this edge should contain the point N (see Fig. 1a). The simplest analysis shows that the line MN cannot be the flow condition for any finite domain of variation of the radius. Let us also note that the stress state cannot be determined at once even by the line NJ. Indeed, in this case the equality

$$k_1 = -k_2, \tag{3.2}$$

resulting in a differential equation for w

$$\frac{d^2w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} = 0,$$

whose solution is the expression

$$w = C_1 \ln \rho + C_2 \tag{3.3}$$

should be satisfied because of the associated flow law in a finite domain.

By virtue of the boundary conditions (3.1) we obtain $w \equiv 0$. Therefore, near the edge $\rho = 1$ a domain of finite dimensions should exist in which the stress state is determined by the point N:

$$m_2 = 0, \ m_1 = -1 + \nu k_1. \tag{3.4}$$

We note the inner boundary of this zone by the radius ρ_2. In this case the equilibrium equation (2.1) has the solution

$$m_1 = -p + C/\rho, \tag{3.5}$$

where C is the constant of integration.

Taking account of (3.4) and (3.5), and the expression (1.4) for k_1, we can obtain the equation

$$\frac{d^2w}{d\rho^2} = \frac{p-1}{\nu} - \frac{C}{\nu\rho},$$

whose solution under the boundary conditions (3.1) has the form

$$w = \frac{(p-1)(\rho-1)}{2\nu} - \frac{C}{\nu}(\rho \ln \rho - \rho + 1). \tag{3.6}$$

For $\rho < \rho_2$ the flow condition should be represented by the line NJ:

$$m_2 - m_1 = 1 + \nu k_2, \ m_1 < 0. \tag{3.7}$$

The condition (3.2) should correspondingly be satisfied, and the displacement is represented by (3.3); the flow condition (3.7) and the equilibrium equation take the form

$$m_2 - m_1 = 1 - \nu C_1/\rho^2, \ \rho dm_1/d\rho = -(p-1) - \nu C_1/\rho^2.$$

If it is considered that plastic flow in the interval $[\rho_2, \xi]$ is characterized by condition (3.7), by taking account of the boundary conditions

for $\rho = \xi \quad m_1 = 0,$

for $\rho = \rho_2 \quad m_2 = 0, \quad m_1 = -1 + \nu C_1/\rho_2^2,$

an equation can be obtained that connects the quantities ρ_2, p, and C_1:

$$(p-1)(\ln \rho_2 - \ln \xi) + \frac{\nu C_1}{2}\left(\frac{1}{\xi^2} + \frac{1}{\rho_2^2}\right) = 1. \tag{3.8}$$

The condition of continuity of the quantities m_1 and $dw/d\rho$ for $\rho = \rho_2$ yields two other equations

$$-p + \frac{C}{\rho_2} = -1 + \frac{\nu C_1}{\rho_2^2}, \quad \frac{p-1}{\nu}(\rho_2 - 1) - \frac{C \ln \rho_2}{\nu} = \frac{C_1}{\rho_2}. \tag{3.9}$$

The system (3.8) and (3.9) reduces to one transcendental equation in ρ_2

$$(p-1)\left[2(1 + \ln \rho_2)(\ln \rho_2 - \ln \xi) + \rho_2(\rho_2 - 1 - \rho_2 \ln \rho_2)\left(\frac{1}{\xi^2} + \frac{1}{\rho_2^2}\right)\right] = 2(1 + \ln \rho_2). \tag{3.10}$$

The rate of displacement of the orifice edge w_0 can afterwards be calculated from the formula

$$\nu w_0 = (p-1)(\rho_2-1)^2/2 - C(\rho_2 \ln \rho_2 - \rho_2 + 1) + \nu C_1(\ln \rho_2 - \ln \xi). \quad (3.11)$$

The condition for applicability of the solution obtained is satisfaction of the inequality (3.7), which can in turn be replaced by the inequality

$$dm_1/d\rho|_{\rho=\xi} \leqslant 0. \quad (3.12)$$

The equality sign in this last expression evidently corresponds for given p to the limit value ξ_{lim} for which the stress state is still characterized by two zones (3.4) and (3.7). Appending this equality in the form

$$p-1 = -\nu C_1/\xi^2_{lim}$$

to the system (3.8) and (3.9), we find

$$\xi_{lim} = \sqrt{\frac{\rho_2(1+\rho_2 \ln \rho_2 - \rho_2)}{1+\ln \rho_2}},$$

where ρ_2 is determined from the equation

$$(p-1)\left[\ln \rho_2 - \ln \sqrt{\frac{\rho_2(1+\rho_2 \ln \rho_2 - \rho_2)}{1+\ln \rho_2}} - \frac{1}{2} + \frac{\rho_2 - 1 - \rho_2 \ln \rho_2}{2\rho_2(1+\ln \rho_2)}\right] = 1.$$

It is evident that if the expression in the square brackets in the last equation is equated to zero, the value $\xi_{lim} = \xi_K$ obtained will correspond to the minimal orifice size for which the solution found above will be true for any quantity p. The dependence of the quantity ξ_{lim} on the load p is shown in Fig. 2. The critical value is $\xi_K = 0.3234$ (dashed line). The results of computations using (3.8)-(3.11) for the dependences of the deflection of the orifice edge and the value of ρ_2 on the acting load are presented in Fig. 3 (lines 1-3 correspond to $\xi = 0.8, 0.6, 0.4$). Here $p_0 = 1 - \ln^{-1}\xi$ is the limit load for a rigidly plastic plate with the same value of ξ.

It is legitimate to assume that in those cases when condition (3.12) is spoiled, still another plastic zone will appear near the orifice, in which the flow condition will correspond to the line JS (as has been shown, the point J cannot be a flow condition for a plate in a finite interval of variation of the radius).

Let the zone boundary correspond to the radius ρ_1 ($\xi < \rho_1 < \rho_2$). In conformity with the associated flow law $k_1 = 0$, which results in an expression for the displacement rate in this zone

$$w = \frac{(w_0 - w_1)(\rho_1 - \rho)}{\rho_1 - \xi} + w_1,$$

where w_1 is the rate of deflection for $\rho = \rho_1$.

Taking account of the flow condition $m_2 = 1 + \nu k_2$ the equilibrium equation can be written in the form $d(\rho m_1)/d\rho = p - 1 + \nu C_3/\rho$, $C_3 = (w_1 - w_0)/(\rho_1 - \xi)$.

Solving the last equation with the equality $m_1 = 0$, taken into account on the boundaries of the interval $[\rho_1, \xi]$, we obtain the expression

$$(p-1)(\rho_1 - \xi) + \nu C_3(\ln \xi - \ln \rho_1) = 0. \quad (3.13)$$

From the continuity conditions for m_1 and $dw/d\rho$, we have for $\rho = \rho_1$

$$\frac{C_1}{\rho_1} = -C_3, \quad (p-1)(\ln \rho_2 - \ln \rho_1) + \frac{\nu C_1}{2}\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right) = 1. \quad (3.14)$$

The system (3.9), (3.13), (3.14) permits determination of the unknown quantities C, C_1, C_3, ρ_1, ρ_2 for given values of p and ξ, after which the rate of displacement of the orifice edge is determined from the formula

$$\nu w_0 = \frac{(p-1)(\rho_2 - 1)^2}{2} - C(\rho_2 \ln \rho_2 - \rho_2 + 1) - \nu C_1(\ln \rho_2 - \ln \rho_1) + \nu C_3(\rho_1 - \xi).$$

A detailed analysis shows that the solution obtained is applicable for $\xi \geqslant \xi_K \xi_1 = 0.0163$. The curve $\xi_{lim} - p$ (see Fig. 2) separates the domains of the two and three-zone schemes of the

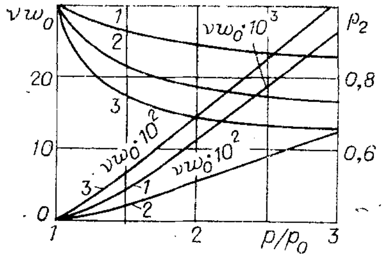

Fig. 3

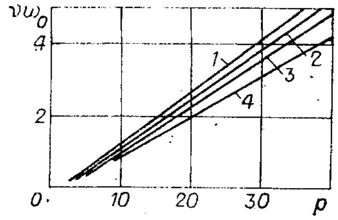

Fig. 4

plate flow. The results of computations of the dependences $\nu w_0 - p$ are presented in Fig. 4 for the values $\xi < \xi_K$ (ξ = 0.1, 0.2, 0.25, 0.3 are the lines 1-4).

One interesting feature of the solutions obtained for viscoplastic deformation problems should be noted in the case of a linear function Φ. The linear dependence of the characteristic rates of deflection on the load (2.5), (2.7) is sufficiently regular for all plates in the presence of one flow mode, however, an analogous dependence in the presence of several zones with moving boundaries is somewhat unexpected. Nevertheless, despite the awkwardness of the analysis, the deviations from the linear dependence are not large in all cases for the known solutions (see [1, 2, 4-6], Figs. 3 and 4), and are remarked only in the domain of load values near the static limit load. For instance, for a circular plate loaded by uniform pressure [4], the deviations from the linear dependence in the whole range of displacement rates do not exceed 1% of the static limit load.

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INVERSE PROBLEM OF MEMBRANE DEFORMATION UNDER CREEP CONDITIONS

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1. Inverse problems of membrane deformation under creep conditions in a given time in a convex surface for minimal energy expenditures occur, for instance, in analyzing technological equipment for pressure treatment of materials in the creep regime [1].

Let us consider a membrane occupying a domain S in the x_1Ox_2 plane that is bounded by the outline γ and is being deformed under the action of external forces q normal to its plane and p_k (k = 1, 2) applied to γ and lying in its plane. The equilibrium equations have the form [2]

$$\frac{\partial\sigma_{kl}}{\partial x_l} = 0 \quad (k = 1, 2), \quad h\sigma_{kl}\frac{\partial^2 w}{\partial x_k \partial x_l} = -q, \tag{1.1}$$

where σ_{kl} (k, l = 1, 2) are stress tensor components, h is the membrane thickness, and w is its deflection. Summation from 1 to 2 is over the repeated subscripts.

The strain tensor components ε_{kl} (k, l = 1, 2) are related to the displacement components u_k (k = 1, 2) in the x_1Ox_2 plane and the deflection w by the following dependences [2]:

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